MATH 579 Exam 2 Solutions

1. Using mathematical induction, prove that for all positive integers n, we have $1+3+5+\cdots+(2n-1) = n^2$.

   For $n = 1$, $1 = 1^2$. Now, using (weak) induction, we assume that $1 + 3 + \cdots + (2n - 1) = n^2$. We add $2(n + 1) - 1 = 2n + 1$ to both sides, concluding $1 + 3 + \cdots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$, as desired.

2. Let $a_0 = a_1 = 1$, and $a_{n+2} = a_{n+1} + 5a_n$ for $n \geq 0$. Prove that $a_n \leq 3^n$ for all $n \geq 0$.

   For $n = 0, 1$, we have $1 \leq 1 = 3^0$ and $1 \leq 3 = 3^1$. Otherwise, using strong induction, we assume that $a_{n+1} \leq 3^{n+1}$ and $a_n \leq 3^n$. We have $a_{n+2} = a_{n+1} + 5a_n \leq 3^{n+1} + 5 \cdot 3^n = (3 + 5)3^n = 8 \cdot 3^n < 9 \cdot 3^n = 3^{n+2}$, as desired.

3. Given $n \in \mathbb{N}$, the *alternating sum* of n is given as the units digit, minus the tens digit, plus the hundreds digit, minus the thousands digit, etc. For example, the alternating sum of $7,904,567$ is $7 - 6 + 5 - 4 + 0 - 9 + 7 = 0$. Prove that n is a multiple of 11 if and only if its alternating sum is a multiple of 11.

   Let $as(n)$ denote the alternating sum of n. It is enough to prove that $n - as(n)$ is a multiple of 11. We use induction on the number of digits of n. If n has one digit, then $n - as(n) = 0$, which is a multiple of 11. Otherwise, we write $n = 10a + b$, for integers a, b, where $a < n$ and $b \in [0, 9]$. We have $as(n) = b - as(a)$, hence $n - as(n) = (10a + b) - b + as(a) = 11a - a + as(a) = 11a - (a - as(a))$. By strong induction, we know that 11 divides $a - as(a)$, and 11 divides $11a$, hence 11 divides their difference $n - as(n)$.

4. Prove that for every triangulated simple polygon, it is possible to color each of its vertices red, blue, or green such that every triangle has its three vertices of different colors.

   We proceed by induction on the number of sides of the polygon. If only three sides, there are only three vertices, so the coloration is obvious. Otherwise, we use the theorem, proved in class, that every triangulated simple polygon has at least one exterior triangle. Removing that exterior triangle gives a polygon with one fewer side, which inherits the triangulation from the original polygon. By the inductive hypothesis, we may color the vertices of the smaller polygon in the desired manner. Replacing our exterior triangle, we see that two of its vertices have been colored already, necessarily with different colors since the edge between them is in some triangle in the smaller polygon. Hence we may color the remaining vertex the third color, as desired.

5. The Fibonacci numbers are defined as $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Let $\phi = \frac{1+\sqrt{5}}{2}$. Prove that $\frac{\phi^n}{3} < F_n < \phi^n$, for all $n \in \mathbb{N}$.

   We begin with the following lemma: $\phi^2 = 1 + \phi$, which is proved via direct calculation $\phi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{1+5+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \phi$. The proof of each inequality will proceed by induction on n. We take care of the base cases via $\frac{\phi^1}{3} \approx 0.5 < 1 < 1.6 \approx \phi^1$ and $\frac{\phi^2}{3} \approx 0.9 < 1 < 2.6 \approx \phi^2$.
   Part 1: We prove $\frac{\phi^n}{3} < F_n$. The base cases were above, so we now assume $F_n > \frac{\phi^n}{3}$ and $F_{n+1} > \frac{\phi^{n+1}}{3}$. We have $F_{n+2} = F_{n+1} + F_n > \frac{\phi^{n+1}}{3} + \frac{\phi^n}{3} = (\phi + 1)\frac{\phi^n}{3} = \phi^2 \frac{\phi^n}{3} = \frac{\phi^{n+2}}{3}$.
   Part 2: We prove $F_n < \phi^n$. The base cases were above, so we now assume $F_n < \phi^n$ and $F_{n+1} < \phi^{n+1}$. We have $F_{n+2} = F_{n+1} + F_n < \phi^{n+1} + \phi^n = (\phi + 1)\phi^n = \phi^2\phi^n = \phi^{n+2}$.